DESCRIPTION OF A SIGNAL BY A DIFFERENTIAL EQUATION OR SEQUENCE OF TAYLOR POLYNOMIALS USING MULTIRESOLUTION ANALYSIS

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Abstract

Multiresolution analysis (MRA) with wavelets approximates not only a signal but also its derivatives, if it is sufficiently smooth and differentiable. This paper reports how a differential equation or a system may be constructed that has signals as solutions. Furthermore, MRA with Haar wavelet is used to approximate the signal by a series of Taylor polynomials. Derivatives of the signal are approximated by MRA packets.

Keywords:

systemanalysis,
system identification,
wavelets,
approximation,
taylor polynomial,
differentialequations,
wavelet packet decomposition

1 Introduction

Systems of natural science are usually described by differential equations. The properties of such systems may be derived from the solutions of the differential equations and the specific initial and boundary conditions as well as the parameter values. However, in nature, one can often measure a property of the system (a signal) without detailed knowledge of the system. The task is to extract information about the system from the signal to find the corresponding differential equation system.

Tools of signal analysis may be used to characterise the system, which has produced the signal. The linear systems of electronics [CD91] have been analysed in this way by Fourier analysis and Laplace transforms [Wil96]. Recent advance has extended this analysis to wavelet transforms [Kai94] [], which analyse signals according to their extent in space and time. Moreover, the wavelet representation allows a freedom in the choice of the basis of functions and requires no longer that the functions of the basis be virtually independent. The latter is shown in the multi resolution analysis (MRA) by wavelets [Dau92].

This contribution shows how the MRA-techniques may be used to construct from a signal an approximated differential equation system, that generates the signal. Moreover, it gives a sequence of Taylor polynomials, with which the signal may be approximated.

2 Derivatives of a Continuous Signal in Wavelet Analysis

Suppose that a signal S is measured at n equally spaced time intervals t_i and gave the values S_i, $(i = 1..n)$. Then the signal may be approximated by a curve:

$$S(t) = \sum_{i=1}^{n} S_i \Phi_i(t) \tag{1}$$

where the functions are

$$\Phi_{i_0}(t) = \begin{cases} 1 & \text{if} \left(t_i - \frac{\Delta t}{2}\right) < t \leq \left(t_i + \frac{\Delta t}{2}\right) \\ 0 & \text{else} \end{cases} \quad \text{and} \quad \Delta t = \frac{(t_{i+1} - t_i)}{2} \tag{2}$$

The signal may be decomposed by

$$S(t) = \sum_{i=1}^{\frac{n}{2}} \left\{ \frac{S_{2i-1} + S_{2i}}{2} \Phi_{i1}(t) \right\} + \sum_{i=1}^{\frac{n}{2}} \left\{ \frac{S_{2i} - S_{2i-1}}{2} \psi_{i1}(t) \right\} \tag{3}$$

$$\text{where } n \text{ is an even natural integer} \tag{4}$$

into two additive terms, of which the first is an average over each two successive data and the second consists of its difference. The new functions are:

$$\phi_{i,1} = \phi_{2i-1,0} + \phi_{2i,0} \qquad \psi_{i,1} = -\phi_{2i-1,0} + \phi_{2i,0} \tag{5}$$

The functions ψ subtracts details from the signal. The coefficients in the detail terms of the signal approximate the derivative with respect to t the finer, the smaller the interval Δt, at which measured values are taken,is, i.e.

$$\lim_{\Delta t \to 0} \left(\frac{S_{i+1} - S_i}{\Delta t} \right)_{S_i} \to \frac{dS}{dt} \tag{6}$$

This implies that the derivative exists. Equation 6 is not necessarily valid, if the values S_i originate from noise. However, for noisy data the averaging and differencing decomposition of equation 3 is still valid.

The above procedure may be repeated and applied to the averaged part, as well as to the differenced part of the signal, to give

$$\begin{aligned} S(t) = & \sum_{i=1}^{\frac{n}{4}} \frac{1}{2} \left(\frac{S_{i-3} + S_{i-2}}{2} + \frac{S_{i-1} + S_i}{2} \right) \phi_{2i}(t) + \sum_{i=1}^{\frac{n}{4}} \frac{1}{2} \left(\frac{S_{i-3} + S_{i-2}}{2} - \frac{S_{i-1} + S_i}{2} \right) \psi^1_{i2}(t) \\ & + \sum_{i=1}^{\frac{n}{4}} \frac{1}{2} \left(\frac{S_{i-3} - S_{i-2}}{2} + \frac{S_{i-1} - S_i}{2} \right) \psi^2_{i2}(t) + \sum_{i=1}^{\frac{n}{4}} \frac{1}{2} \left(\frac{S_{i-3} - S_{i-2}}{2} - \frac{S_{i-1} - S_i}{2} \right) \psi^3_{i2}(t) \end{aligned} \tag{7}$$

where $n = 4 \cdot m$ and m is a positive integer and

$$\phi_{i2} = \phi_{2i-1,1} + \phi_{i,1} \tag{8}$$
$$\psi^2_{i2} = \psi_{2i-1,1} + \psi_{i,1} \tag{9}$$
$$\psi^1_{i2} = -\phi_{2i-1,1} + \phi_{i,1} \tag{10}$$
$$\psi^3_{i2} = -\psi_{2i-1,1} + \psi_{i,1} \tag{11}$$

The coefficients in the above sum are generated by introducing an average operation $\hat{a}$ and a differencing operation $\hat{d}$

$$S(t) = \sum_{i=1}^{\frac{n}{4}} (\hat{a} \cdot \hat{a})\, S_i \cdot \phi_{i2}(t) \quad + \quad \sum_{i=1}^{\frac{n}{4}} \left(\hat{d} \cdot \hat{a}\right) S_i \cdot \psi^1_{i2}(t) \tag{12}$$
$$+ \sum_{i=1}^{\frac{n}{4}} \left(\hat{a} \cdot \hat{d}\right) S_i \cdot \psi^2_{i2}(t) \quad + \quad \sum_{i=1}^{\frac{n}{4}} \left(\hat{d} \cdot \hat{d}\right) S_i \cdot \psi^3_{i2}(t) \tag{13}$$

In equation 6 there are two possibilities of obtaining an approximation of the derivative of S. Either by applying the averaging operation followed by a differencing operation $(\hat{d} \cdot \hat{a})$ to the signal or by applying the differencing operation onto the averaging operation $(\hat{a}\cdot\hat{d})$ on the signal. The difference between the second and the third term of eq. 7 gives

$$(\hat{d}\cdot\hat{a})S_{4i} - (\hat{a}\cdot\hat{d})S_{4i} = S_{4i-2} - S_{4i-1} \tag{14}$$

$$\text{where } (i = 1\ldots m) \tag{15}$$

In eq. 7 the new coefficients are calculated always from four successive values of S_i . For simplicity, the first four values may be used to refer differencing operation $\hat{d}$ to its corresponding intervals Δt at the times t_i, $(i = 1\ldots 4)$ in the terms of eq. 7 (see figure 1)

$$\frac{\hat{a}\cdot\hat{d}}{\Delta t} = \tfrac{1}{2}\tfrac{S_4-S_3}{t_4-t_3} + \tfrac{1}{2}\tfrac{S_2-S_1}{t_2-t_1} = \frac{1}{2}\left(\frac{S_4-S_3}{\Delta t} + \frac{S_2-S_1}{\Delta t}\right) \tag{16}$$

$$\frac{\hat{d}\cdot\hat{a}}{2\Delta t} = \frac{\frac{S_4+S_3}{2} - \frac{S_2+S_1}{2}}{\frac{t_4+t_3}{2} - \frac{t_2+t_1}{2}} = \frac{1}{2}\left(\frac{S_4-S_2}{2\Delta t} + \frac{S_3-S_1}{2\Delta t}\right) \tag{17}$$

$$\frac{S_3-S_2}{t_3-t_2} = \frac{S_3-S_2}{\Delta t} \tag{18}$$

In the limit $\Delta t \to 0$, equations $16 - 18$ tend toward the derivative with respect to t in the interval t_1 to t_4. For eq. 14, the limit $\Delta t \to 0$ reads

$$\lim_{\Delta t\to 0}\left(\frac{\hat{d}\cdot\hat{a}}{2\Delta t}S_i - \frac{\hat{a}\cdot\hat{d}}{\Delta t}S_i = \frac{\Delta S_i}{\Delta t}\right) \to \left(2\frac{dS}{dt}\right)\bigg|_{S_i} - \frac{dS}{dt}\bigg|_{S_i} = \frac{dS}{dt}\bigg|_{S_i} \tag{19}$$

or if $S_3 - S_2$ is considered as a differencing operation shifted by one in the index then the limit reads

$$\hat{d}\cdot\hat{a} - \hat{a}\cdot\hat{d} = \hat{d} \tag{20}$$

i.e. $\hat{a}$ may leave $\hat{d}$ invariant.

The next higher order of signal decomposition is obtained by applying the operation of differencing and averaging to the coefficients of eq. 7. Thus, the expressions of multiple averaging and differencing operations result in e.g.

$$\hat{d}\cdot\hat{a}\cdot\hat{d}\cdot\hat{a} \tag{21}$$

On substitution of eq. 12 into eq. 21 and rearranging to shift the averaging operation to the left, from

$$\hat{d} \cdot \hat{a} \cdot \hat{d} \cdot \hat{a} = (\hat{a} \cdot \hat{d} + \hat{d}) \cdot (\hat{a} \cdot \hat{d} + \hat{d}) \tag{22}$$

one obtains finally:

$$= (\hat{a} \cdot \hat{a} + 3 \cdot \hat{a} + 2) \cdot \hat{d} \cdot \hat{d} \tag{23}$$

In the limit $\Delta t \to 0$ (infinitely dense time sequence), the averaging operations contract to a constant k i.e.

$$\lim_{\Delta t \to 0} \left(\frac{\hat{d} \cdot \hat{a} \ \cdot \ \hat{d} \cdot \hat{a}}{\Delta t \cdot \Delta t} \right) = k \cdot \frac{d}{dt} \left(\frac{d}{dt} \right) \tag{24}$$

Similarly for the other expressions. This leads to the

Proposition (I)

Suppose the unique limit for $\Delta t \to 0$ exists for a sequence of differencing and averaging operations of equally dense spaced values in t (time), then this limit is proportional to the n-th order differential-quotient in the interval, on which the sequence of differencing and averaging operations is acting, and n is the number of differencing operations in the sequence.

This proposition may be verified by iterating the example given in eq.'s 7 to 24.

Since the expansion of a function in a power series (Taylor-series) has derivatives as coefficients, approximations to these derivatives may be obtained from the differencing operations of the signal.

In an interval comprising one averaging $\hat{a}$ and one differencing operation $\hat{d}$, the power series

$$y = y_m + y_m^{(1)} \cdot (t - t_m) \quad \text{at} \quad t_m = \frac{(t_{2i} - t_{2i-1})}{2} \tag{25}$$

has the coefficients :

$$y_m = \hat{a} S_i = \frac{(S_{2i} + S_{2i-1})}{2} \quad \text{and} \quad y_m^{(1)} = \hat{d} S_i = \frac{\frac{(S_{2i} - S_{2i-1})}{2}}{\Delta t}$$

where $i = 1, 2, \ldots, \frac{n}{2}$

At the power of two, the polynomial expands to

$$y = y_m + y_m^{(1)}(t - t_m) + \frac{y_m^{(2)}}{2!}(t - t_m)^2 + \varepsilon(t) \tag{26}$$

where $\varepsilon(t)$ is the residual term

The averaging and differencing operations yield the coefficients (see eq. 16, eq. 17 and eq. 23) for $\hat{a}\hat{d}$, $\hat{d}\hat{a}$ and $\hat{d}\hat{d}$ and the arithmetic mean $(\hat{a}\hat{a}) \cdot S$, respectively. Where as, in the previous case, y_m and $y_m^{(2)}$ are uniquely defined, there are in this case two possibilities of approximating $y_m^{(1)}$, i.e. by $\hat{d}\hat{a}$ or by $\hat{a}\hat{d}$. Also, eq. 18 (or in general eq. 20) delivers $y_m^{(1)}$, which should usually yield the better approximation of $y_m^{(1)}$ since it evaluates the derivative between t_2 and t_3 (see figure 1 and eq. 16 to eq. 18) at a shorter distance to t_m. This assigns an additional significance to eq. 20 because $\hat{a}\hat{d} - \hat{d}\hat{a}$ is a good approximation of $y_m^{(1)}$. When the power series is expanded to higher degrees, the overestimation of its coefficients by the approximations of averaging and differencing operations at a down sampling level grows. An estimate of the error of the term $\varepsilon(t)$ in eq.(26) can be obtained by evaluating eq. 26 at $t = t_1$ to t_4 in the scheme of fig. 1 and calculating the departure from the signal value S. In the limit where, $\hat{a}\hat{d}$, $\hat{d}\hat{a}$ and $\hat{d}$ all are proportional to the differential quotient $\varepsilon(t)$ should be negligible at the signal points.

If a non-normalised basis is choosen e.g., by replacing ψ_{i2}^1 in equation 12 by $2 \cdot \psi_{i2}^1$, one obtains a commuting relation $\hat{d} \cdot \hat{a} - \hat{a} \cdot \hat{d} = 0$ for equation 20.

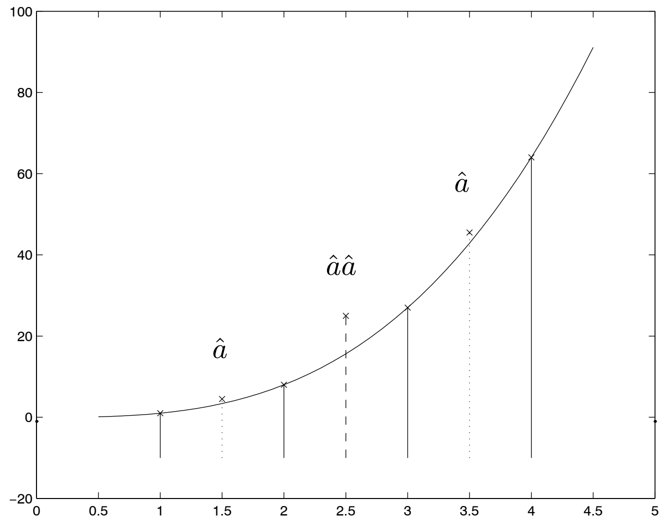

Figure 1: *Scheme to approximate the signal by a power series of an incresing degree. Four successive values of the signal at t_1 to t_4 and (as doted lines) the values obtained by the averaging operations $\hat{a}$ and $\hat{a}\hat{a}$ at the corresponding values t are shown*

3 Multi-resolution Analysis (MRA) with Haar Wavelets as Basis

Multi-resolution Analysis decomposes a signal S into a down-scaled signal and the subtracted details (see eq. 3) i.e.

$$S = (\hat{a} + \hat{d})S = \hat{a} \cdot S + \hat{d} \cdot S \tag{27}$$

where in the case of Haar Wavelets, which are used thoughout this report, $\hat{a}$ and $\hat{d}$ are the averaging and differencing operations of the previous section. A renewed decomposition of the averaged and differenced signal gives a down-scaled signal, from which more details have been subtracted (see eq. 7)

$$\begin{aligned} S &= (\hat{a} + \hat{d}) \cdot \hat{a} \cdot S + (\hat{a} + \hat{d}) \cdot \hat{d} \cdot S \\ &= \left(\hat{a} \cdot \hat{a} + \hat{d} \cdot \hat{a} + \hat{a} \cdot \hat{d} + \hat{d} \cdot \hat{d}\right) \cdot S \end{aligned} \tag{28}$$

This procedure can be continued, until the sequence of data of S is used up. At the third level, one obtains

$$\begin{aligned} S = (\hat{a} \cdot \hat{a} \cdot \hat{a} &+ \hat{a} \cdot \hat{a} \cdot \hat{d} + \hat{a} \cdot \hat{d} \cdot \hat{a} + \hat{d} \cdot \hat{d} \cdot \hat{a} + \cdots \\ &+ \hat{d} \cdot \hat{a} \cdot \hat{a} + \hat{d} \cdot \hat{a} \cdot \hat{d} + \hat{a} \cdot \hat{d} \cdot \hat{d} + \hat{d} \cdot \hat{d} \cdot \hat{d}) \cdot S \end{aligned} \tag{29}$$

If proposition (I) is applied to this decomposed signal:

1. the averaged down-sampled signal is $(\hat{a} \cdot \hat{a} \cdot \hat{a}) \cdot S$
2. the details with a first differential-quotient as the limit are $(\hat{a} \cdot \hat{a} \cdot \hat{d} + \hat{a} \cdot \hat{d} \cdot \hat{a} + \hat{a} \cdot \hat{a} \cdot \hat{d}) \cdot S$
3. the details with a second differential quotient as the limit are $(\hat{d} \cdot \hat{d} \cdot \hat{a} + \hat{d} \cdot \hat{a} \cdot \hat{d} + \hat{a} \cdot \hat{d} \cdot \hat{d}) \cdot S$
4. the detail with a third differential-quotient as the limit is $(\hat{d} \cdot \hat{d} \cdot \hat{d}) \cdot S$

Proposition (I) demands for the details in section (2): they should be proportional to each other in the limit $\Delta t \to 0$ and to the first differential quotient of the signal, if the samples are taken in a sufficiently dense sequence.

$$(\hat{a} \cdot \hat{a} \cdot \hat{d}) \cdot S \simeq k_1(\hat{a} \cdot \hat{d} \cdot \hat{a}) \cdot S \simeq k_2(\hat{a} \cdot \hat{a} \cdot \hat{d}) \cdot S \simeq k_d \cdot \frac{dS}{dt} \tag{30}$$

where k_i are proportionality factors

The same applies to the second differential quotient, i.e.

$$(\hat{d} \cdot \hat{d} \cdot \hat{a}) \cdot S \simeq k_3(\hat{d} \cdot \hat{a} \cdot \hat{d}) \cdot S \simeq k_4(\hat{a} \cdot \hat{d} \cdot \hat{d}) \cdot S \simeq k_{dd} \cdot \frac{d}{dt}\left(\frac{dS}{dt}\right) \tag{31}$$

Each level of down-sampling of the signal yields in its details approximations to the derivatives with respect to the sample sequence up to the order of the level of down-sampling. A detail of the decomposition of the signal, called packet, may be identified by the sequence of averaging and differencing operations, which generated it, e.g. in eq. 29 $\hat{a}\hat{a}\hat{d}$: is the name of the packet $(\hat{a}\hat{a}\hat{d}) \cdot S$.

4 Differential-equation-system obtained by the MRA

Since the details of the MRA may be considered as approximations of derivatives of the signal S, a differential equation system may be constructed with the signal as one solution.

$$F\left(S, \partial_s, \partial_s^2, \partial_s^3, \cdots, \partial_s^m\right) = 0 \tag{32}$$

where F is a function and ∂_s^m the m-th derivative of the signal S with respect to the sample sequence (i.e. time). The m-th order differential equation (32) may be casted into a differential equation system. The function F is restricted to intervals of the down-sampled level of the signal.

$$\begin{aligned}
\partial_s &= s_1 \\
\partial_{s_1} &= s_2 \\
&\vdots \\
\partial_{s_{m-2}} &= s_{m-1} \\
\partial_s^m &= f(S, s_1, s_2, \cdots, s_{m-1})
\end{aligned} \tag{33}$$

where the function f is obtained if equation 32 is solved for ∂_s^m, and ∂_{s_j} is the derivative of s_j with respect to time.

The signal S defines a curve in the hyperspace spanned by the s_i and S, which lies on the hyperplane f. The curve is defined solely on the hyperplane f given by the points (derivations) $\{S_i\}$ at each t_i. Otherwise the hyperplane may be chosen freely. The hypersurface f (eq. 33) itself may be approximated by wavelets in an $m+1$-dimensional space where the values of the function f are coefficients of the wavelet basis. The curve on the hypersurface is represented by some of these coefficients, whereas all other coefficients may be chosen arbitrarily.

5 An experimental example

A signal from an experimental chemical system may be analysed by the methods described to verify the above analysis. The signal in figure 2 was recorded by a spectrophotometer and shows the variation in time of the concentration of the yellow ion Ce^{4+}. The chemical reaction system produces non-linear chemical oscillations in time.

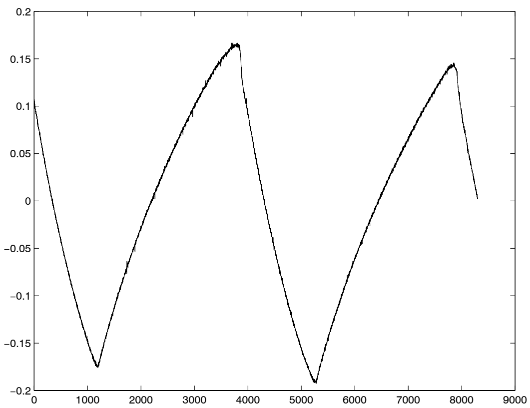

Figure 2: *Spectrophotometric record of the concentration of the yellow ion* Ce^{4+} *versus time in a chemical reaction system displaying non-linear oscillations. The record is used as a test signal in the analysis.The record is digitalised by a voltmeter that has a sensitivity of* ± 0.1 *mV in a range of* ± 200 *mV. The signal is the least noisy at this level.*

The program package WaveLab [Mal98] and [Don97] for computing is used to apply MRA to the signal S in figure 2. It is based on the computing environment of the MATLAB language [Inc97]. The computation is executed on a workstation (SUN sparc Solaris 2.5.1). From the $\sim$ 60000 samples in the record, 8192 were selected for the analysis (see figure 2). The program calculates for the Haar basis and other wavelet basis at every level of down-sampling the decomposition of the signal into all its details and averaged signals.

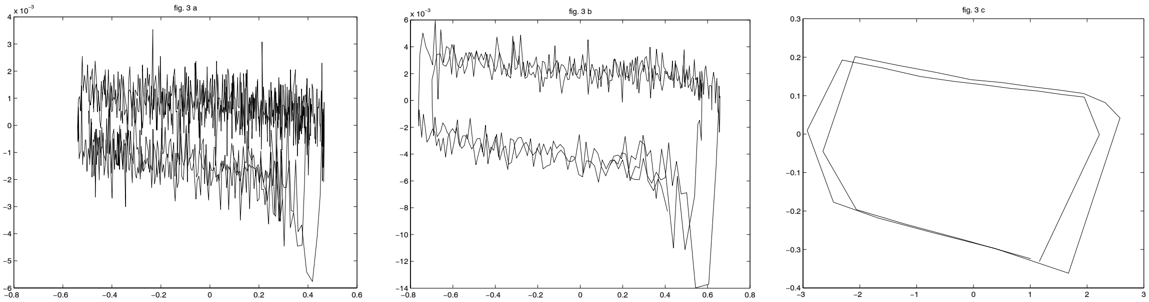

Figure 3: *Phase-portrait of different levels(3,4 and 8 resp.,left to right) of the down-sampling of the Multiresolution Analysis of the averaged signal S horizontal versus the averaged signal vertical axis, to which the differencing operation has been applied once. At down-sample level 8 (right panel), the noise is so filtered that the limit cycle (closed curve) is visible.*

In figure 3, phase portraits of selected levels of down-sampling display the influence of the noise on the differencing operation $\hat{d}$. From the down-sampling level five and onwards, the differencing operations approximates the differential-quotient of S with respect to t. However, at down-sampling levels greater than nine, the interval steps are too coarse, so that only a very poor approximation is obtained. Therefore, the proposition (I) is expected only to be satisfied reasonably well in the range of levels, at which the approximation of the differential quotient is reasonable.

In figure 4, a selection of packets of the MRA of the signal of figure 2 is plotted against each other. They originate from the same order of differencing operations. At down-sampling level 2, where the differencing operations of the same order appear at first, the signal is too noisy to give good approximations of its differential-quotient (panel (a)). However, at level 8 (panel (b) to (d)), after a few averaging steps, there is a good correlation between different packets of first order derivatives. The panels ((b) to (d)) demonstrate, how the averaging operation influences the correlation, when averaging is performed before or after the differencing step. When averaging succeed differencing (panel (d)), the points are more scattered than when averaging precedes the differencing step (panel (b)) . Evidently, the correlation is dependent on the type of noise. The panels (e) and (f) show the relation for second and third order derivatives. Since these derivatives are spikes, they are mostly found close to zero and are not expected give a close correlation because of their critical behaviour towards differentiation in the spike region.

As expected, the more differencing steps are involved in the formation of a packet, the poorer the correlation is (compare panels (d) to (f)). Nevertheless, the solid line in figure 4, which gives the theoretical relation, is fairly well approximated by the experimental points.

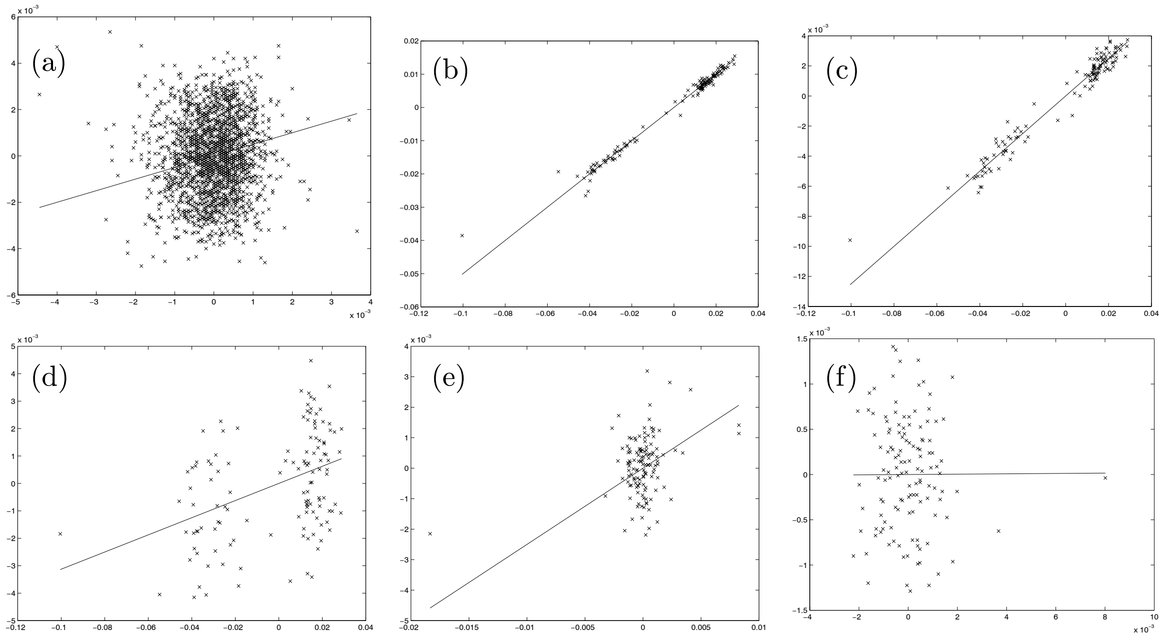

Figure 4: *Test of the proposition that the averaging operation* $\hat{a}$ *leaves the differencing operation* $\hat{d}$ *invariant at the down-sampling-levels 2 and 8. At level 2 the proposition is invalid because of noise where as at level 8 it is fairly well valid even at three differencing steps. The solid line represents the expected theoretical curve of the experimental signal. In panel (a) the packet* $\hat{a}\hat{d}$ *is plotted versus* $\hat{d}\hat{a}$*, in panel (b)* $\hat{a}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$*, in panel (c)* $\hat{a}\hat{a}\hat{a}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}$*, in panel (d)* $\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{d}\hat{a}\hat{a}$*, versus* $\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$*. In panel(e)* $\hat{a}\hat{d}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$ *versus* $\hat{d}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$ *and in panel (f)* $\hat{a}\hat{a}\hat{a}\hat{d}\hat{d}\hat{d}\hat{a}\hat{a}$ *versus* $\hat{d}\hat{d}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$*.*

A wavelet-analysis may reconstruct signals from details of their decomposition. A signal can be reconstructed by a specific selection of packets of a MRA decomposition [Che95] or by all packets in a given level of down-sampling. This allows a modification of the signal at the packet level and its reconstruction. Ideally all packets of the same order of differencing and same level of down-sampling should give a straight line through the origin, when they are plotted as in figure 4, since they should approach the same differential quotient. This could, e.g. in figure 4, be forced by setting the all ordinate values of the same differencing step (in panels ((b) to (d)):

$$\hat{a}\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\,, \quad \hat{a}\hat{a}\hat{a}\hat{d}\hat{a}a\hat{\hat{a}}\hat{a} \quad \text{and} \quad \hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{d}\hat{a}\hat{a} \quad \text{etc.} \tag{34}$$

to their value on the theoretical curve. Then, proposition (I) would be valid. The abscissa still has to be defined. One could, as in figure 4, choose a packet like $\hat{d}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}\hat{a}$ as abscissa. One may select other references also, as eg. the weighted mean of all packets of the same order of differencing at the same down-sampling level. As weights, the factors k of eq. (30) may be chosen. In figure 4, the means with these factors are used as reference packets.

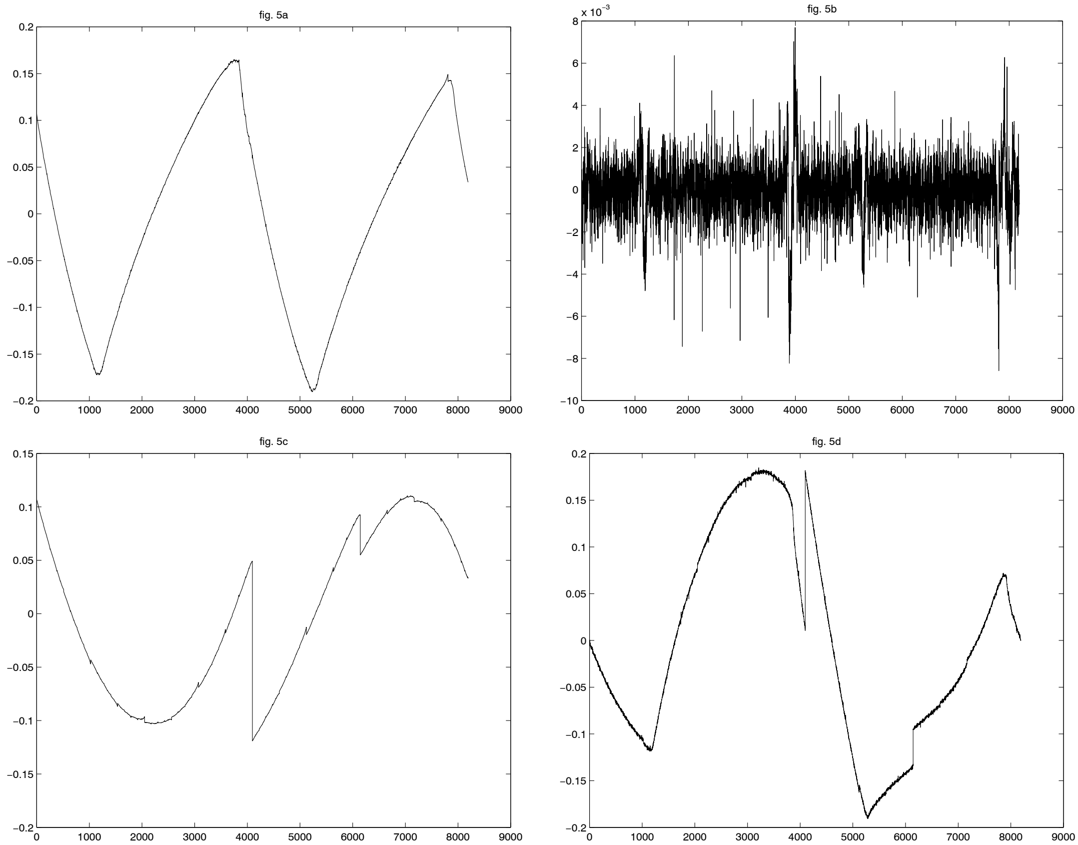

Figure 5: *Reconstructed signal from modified MRA packets and their difference from the original signal of figure 2. Upper panels: at level 8 of down-sampling, proposition (I) has been forced to be valid for all packets by taking a weighted mean as reference over the packets of the same order of derivation. Lower panels: instead of level 8, the deepest level of down-sampling (i.e. here 13) is chosen.*

Reconstruction of a signal from packets of a MRA is a powerful tool to modify the signals specifically or to extract properties from them. In the present case, the packets of one down-sampling level are modified, so that the proposition (I) is satisfied. Reconstruction of the signal from that modified level reproduces the correct overall appearance, as long as the modifications are small. The modified packets should approximate the differential-quotients of the signal.

In figure 5, the upper and lower left panels display the reconstructed signal for two selected levels of down-sampling (level 8 and the lowest level 13), at which the packets have been modified to meet the proposition (I). The reconstructed signal from the level 8 (see upper left panel) resembles the original of figure 2. Apparently the modification does not affect over all shape of the signal. The difference between the two signals (see upper right panel) is small, about $\sim 1\%$ for level 8 and resembles noise. If, at the lowest level (here 13) of down-sampling, the proposition (I) is forced to be valid, the reconstructed signal is usually distorted and shows discontinuities. This is seen in lower left panel of figure 5, since the signal has regions, in which higher derivatives are critical. The difference between the reconstructed and original signal (see lower right panel) is considerable in this case.

The construction of a system of differential-equations producing the signal is in the present case simplified, because the dimension (order) of the differential equation system is considered to be two. On the other hand, difficulties arise in the ranges, where there are sharp changes in the curves, since the first differential quotients of the signal resemble a rectangular curve and the second and higher derivatives are spikes. Nevertheless, figure 6 illustrates that a relation between the signal and its derivatives may be represented by a curve in the three dimensional space spanned by the signal and the first and second derivatives. Any function enclosing this curve on its hypersurface delivers a differential equation system (see eq. 33) which has the signal as a solution. In particular, the series B to D in figure 6 display how the original graph which looks like a cloud of points, can be transformed by applying the proposition (I) at the level 8 and reconstruction of the signal (see figure 6 D) to a graph which demonstrates clearly that the curve $f(S_0, S_1)$ exists.

The obtained descriptive differential equation system expresses some aspects of the chemical system, which emitted the signal, in this case it describes the limit cycle (see fig. 6), which generates the non-linear oscillation. The approach to representation of the signal by a differential equation system leaves much freedom in the completion of the differential system (i.e. the surface in fig. 6 enclosing it).

On the other hand, an expansion in a power (Taylor) series, which use also local differential quotients to approximate the signal, leaves no freedom. From proposition (I) one obtains the coefficients of a Taylor polynomial for each interval of a packet of a down sampling level (see equation 25 and 26). The power series describes the relation between the coefficients of the signal in an interval given by the packet. The term $\varepsilon(t)$ in equation 26 gives a measure of the approximation and may be applied to find a best approximation by taking intervals from different packets. In the right panel of fig. 7, the norm (ε) of the four levels 3 to 6 over all coefficients in the packets is evaluated. In an interval, that packet is selected to construct the Taylor polynomial, for which the norm of $\varepsilon < \varepsilon_{max}$ at the lowest down sampling level

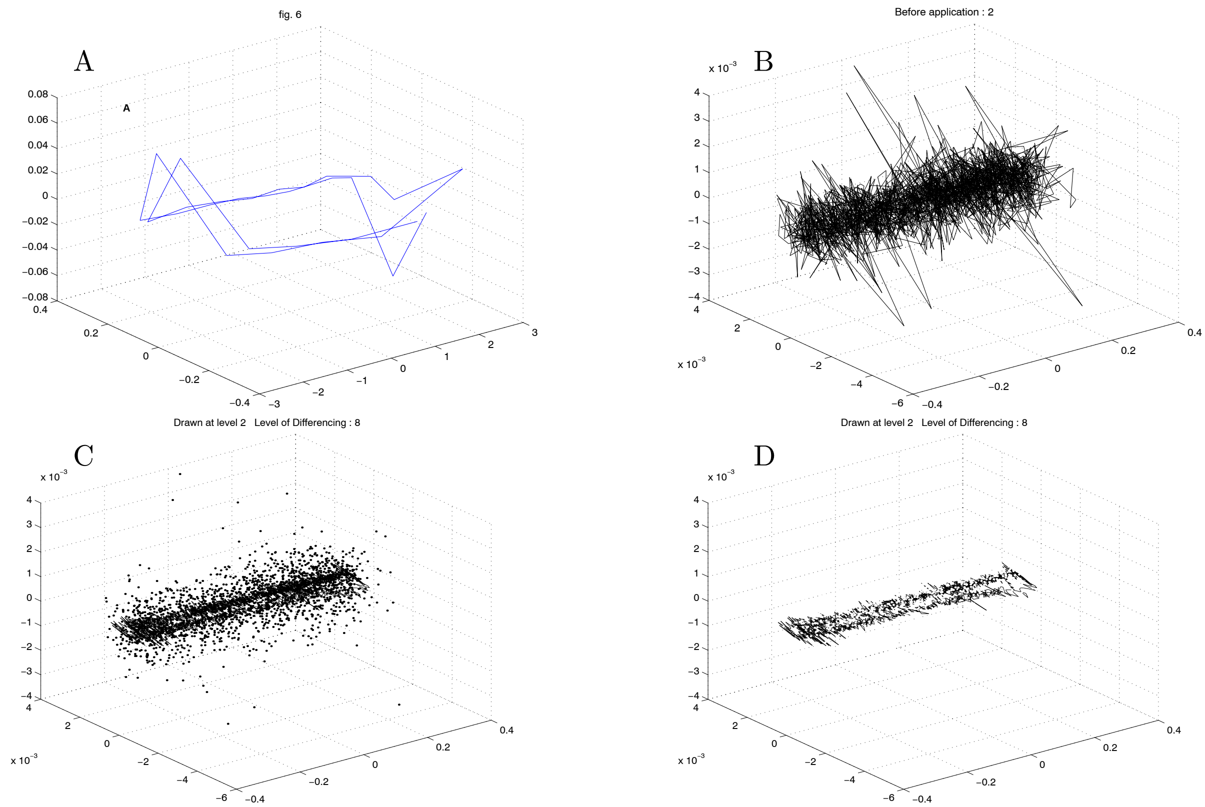

Figure 6: *Signal of figure 2 as a solution of a differential-equation-system ($\frac{d}{dt}S_0 = S_1$, $\frac{d}{dt}S_1 = f(S_0, S_1)$ where $S_0 = \hat{a}\hat{a}S$, $S_1 = \hat{d}\hat{a}S$ and $\frac{d}{dt}S_1 = \hat{d}\hat{d}S$) The axis for S_0 points to the front, for S_1 towards right and for $f(S_0, S_1)$ upwards. Plotted are the coefficients of the wavelet packets (A) at level 8 of down-sampling and (B) at level 2. In (C) and (D), the proposition (I) has been forced to be valid for all packets at level 8 and the reconstructed signal at level 2 is displayed along with (C) or without (D) the original coefficients (see (B)) as dots.*

6 Discussion of the signal analysis

The experimental signal of figure 2 is a time sequence of noisy data. Wavelet analysis not only subtracts noise effectively from the signal into the details [Mal98] but also approximates the differential-quotient of the signal, which figure 4 demonstrates. The suppression of noise is mainly due to the averaging of the signal by the wavelet-analysis.

There are more efficient methods to remove noise from the signal than the one described above [DI94]. Here is only the validity of proposition (I) tested, which should not be fulfilled for noise. Figure 4a demonstrates that at that level no linear relation exists (as demanded by proposition (I)) between the data in the packets compared. However, the more the signal is smoothed by averaging, the more it approaches differentiability and the linear relationship of the proposition

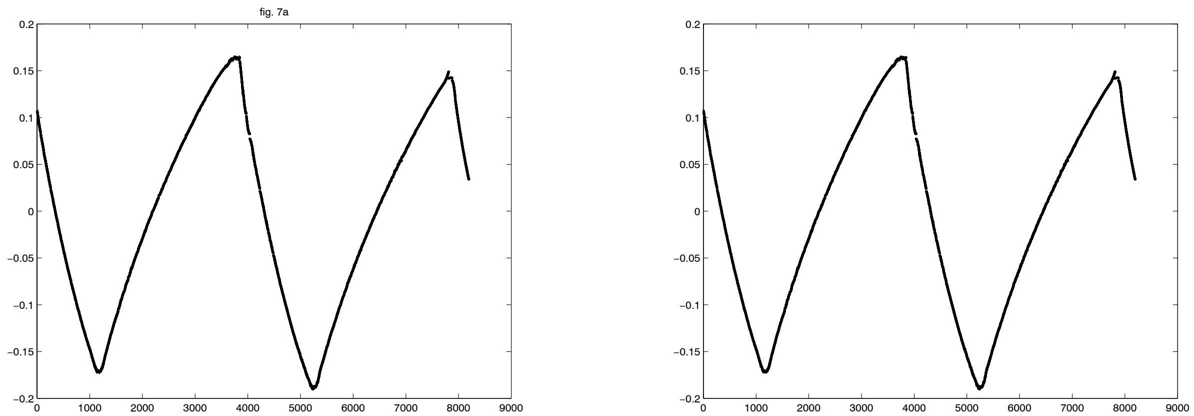

Figure 7: *Approximation of the signal by a series of Taylor polynomials. Left panel: At the down-sampling level 5 in each interval of the MRA packets, the signal is drawn from a Taylor polynomial obtained from the derivatives in that interval. Right panel: as the left panel, except that from the down sampling levels 3 to 6, the best approximation according to a minimal* $\varepsilon_{max} = 0.001$ *in equation 26 is taken as the interval for the Taylor polynomial drawn.*

(I) begins to appear (see figure 4b–e. If the signal is a differentiable curve, proposition (I) should be fulfilled. However, Multi-scale Wavelet Analysis also can forces proposition (I) to hold, because for a given level of down-sampling, all packets of the same level of differencing operation may be set proportional to a reference packet representing the derivation of the signal. There are many ways of constructing the reference packet. The most averaged packet before the first differencing operation may be selected, or a weighted sum of all packets of the same differencing order and level of down-sampling may be chosen. There are also different ways of assigning proportionality factors to the packets. One is to look for the best fit of the original packet to the reference packet. This would yield the best reconstruction of the signal under the condition of the proposition (I). Another is to use theoretical factors and to adapt the reference packets to them. This yields a reconstructed signal, which fulfils the proposition (I), and approximates in addition the differentiability of the signal. The theoretical factors may be calculated or obtained by a Multi-resolution-analysis of the curve x^n.

In figure 5, the reconstructed (synthesised) signal is displayed, which has been modified to fulfil the proposition (I). The signal looks very similar to the original one and the difference between the reconstructed and the original resembles the noise. The lower the level of down-sampling, at which proposition (I) is forced to hold, the more restrictive it is. At the lowest level the signal is reconstructed by only a few values (i.e. the number of derivatives plus one). Complex signals cannot be reconstructed in their details from the lowest level of down-sampling, since usually there are quite large deviations from proposition (I), so that many details have vanished by the forcing of proposition (I) at that level.

In figure 6D, the curve was obtained by reconstruction of the signal from level 8, at which the proposition (I) was forced to hold. This curve is still noisy. The noise may probably be reduced by the techniques of smoothing and the fact is, that at level 8 there are 8 derivatives, of which only two should be independent. Nevertheless, figure 6 shows that for the reproduction of the signal by the differential equation system, only its coefficients are needed. The other coefficients of the surface should not be random but sooner give a smooth (i.e. differentiable) surface, which may satisfy the proposition (I). Variation of the parameters or the initial conditions in the experimental system results in different signals and in additional curves in the diagram of figure 6 and may deliver additional coefficients to characterise the surface and hence the differential equation system in more detail. The differential equation system is of a descriptive form. A coordinate transform, which converts its generalised coordinates into real variables, is needed to obtain a system, which may have a reality in nature. Usually, such a transform will not change the major properties of the system (e.g. a limit cycle remains a limit cycle, but assumes a different form).

The approximation of the signal by a series of Taylor polynomials (see fig. 7) requires that it is to be rather smooth (see [Mal98] chapter 6, Lipschitz regularity). For a basis of Haar wavelets, the series of Taylor polynomials is exact at the first level of down-sampling and reproduces the signal. At a higher levels of down-sampling, the signal may be approximated by a Taylor polynomial only, since the 2^k values of the polynomial are fixed by its k+1 coefficients. The error ϵ (see eq. 26) measures, how well the polynomial approximates the signal. In the left panel of figure 7, a reconstruction of the signal by a series of Taylor polynomials from down-sampling level 5 is displayed, i.e. each of the successive 32 values of the signal are represented by a Taylor polynomial of the order 5. Although the signal is compressed by a factor 32:5, it resembles the original still.

A further compression is obtained, if a lower limit of the error ϵ is chosen and if the highest down-sampling level, for which this limit is valid, is used for the Taylor polynomial to approximate the signal in the corresponding interval. On the other hand, the intervals may be considered analogous to rhythms of a piece of music, in which the notes are represented by coefficients of the Taylor polynomials. The right panel of Fig. 7 gives an example of this view with a basic rhythm of 8 times the sampling rate. The "notes" may be described by a finite number of signs of a duration of 1, 2, 4 and 8 times the basic rhythm, and an intensity given by the coefficients of the Taylor polynomials (for further details of coding the intensity see [Mal98] Chapter XI). If a grammar exists for these signs, then a class of signals may be described by this technique, since music is a subset of all sounds.

A further consequence of proposition (I) is that if it is valid for a given level of down-sampling, then it holds also for all levels above that level.

7 Outlook

This contribution describes an application of the proposition (I) with multi-resolution analysis on Haar wavelets. The proposition is not restricted to Haar wavelets but should be valid for other types of wavelets too. Besides, it could be extended to pictures of two or higher dimensions. In the example, the proposition (I) is used to extract derivatives from a signal [Kra99]. In addition, it may serve as a measure, (e.g. in the "best basis" method [Coi92] of the signal reconstruction or in other methods). Here, the standard deviation in the panels of figure 4 may be used as a measure of "entropy". The signal is then reconstructed by the packets, for which the proposition (I) is optimally fulfilled. On the other hand, the proposition may be applied to find a better basis (e.g. by the methods of wavelets dictionaries [Che95] in which it holds optimal. Also, the proposition is closely related to an algebraic structure, which transforms (rotates) subspaces of MRA into eachother [Kai92]. This transform seems to be related to differentiation and to a generalized form of laws.

In the example the signal is assumed to be produced by a differential equation system of the second order. This implies that derivatives of higher than the second order should be dependent upon those of the lower order. It restricts both proposition (I) and the independence of the packets in the multi-resolution analysis further, (e.g. in the extreme but useless case of the lowest level, only three values would generate all other packets of the MRA). That illustrates the importance of the determination of the order of the expected differential equation system. The order is the dimension of the surface enclosing the curve in figure 6.

Besides, the description of the signal by a differential equation or its corresponding system one by a series of Taylor polynomials is shown. In the former case, the signal is generated in such a way that its value and derivative are given by functions at a specific time. In the latter case, the derivatives may be a set of discrete numbers in definite time intervals. This allows the presentation of signals, which are not easily accessible using regular functions, and stresses the independence of the time intervals. In this connection, it should be mentioned that at the lowest level of down sampling of a MRA, there is only one time interval, i.e. the duration of the signal and the coefficients apply all to this range.

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